Exercises

Derivatives and extrema of multivariate functions – Solutions

Exercise 1. 1.

$$f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\frac{\partial f}{\partial x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)^{-1/2} \cdot 2x_i = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad i = 1, 2, 3;$$

$$\nabla f = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

2. $f(x_1, x_2, x_3) = \sin x_1 \cdot \cos x_2 \cdot \sqrt{x_3}$

$$\frac{\partial f}{\partial x_1} = \cos x_1 \cdot \cos x_2 \cdot \sqrt{x_3}$$

$$\frac{\partial f}{\partial x_2} = -\sin x_1 \cdot \sin x_2 \cdot \sqrt{x_3}$$

$$\frac{\partial f}{\partial x_3} = \frac{1}{2} \sin x_1 \cdot \cos x_2 \cdot \frac{1}{\sqrt{x_3}}$$

$$\nabla f = \begin{pmatrix} \cos x_1 \cdot \cos x_2 \cdot \sqrt{x_3} \\ -\sin x_1 \cdot \sin x_2 \cdot \sqrt{x_3} \\ \frac{1}{2} \sin x_1 \cdot \cos x_2 \cdot \frac{1}{\sqrt{x_3}} \end{pmatrix}$$

3.

$$f(x_1, x_2, x_3) = 2x_1^2 \ln(x_2^2) + e^{x_3^2} \cdot \sin x_1$$

$$\frac{\partial f}{\partial x_1} = 4x_1 \ln(x_2^2) + e^{x_3^2} \cos x_1$$

$$\frac{\partial f}{\partial x_2} = 2x_1^2 \frac{2x_2}{x_2^2} = 4\frac{x_1^2}{x_2}$$

$$\frac{\partial f}{\partial x_3} = 2x_3 e^{x_3^2} \sin x_1$$

$$\nabla f = \begin{pmatrix} 4x_1 \ln(x_2^2) + e^{x_3^2} \cos x_1 \\ 4\frac{x_1^2}{x_2} \\ 2x_3 e^{x_3^2} \sin x_1 \end{pmatrix}$$

4.

$$f(x_1,\, x_2,\, x_3) = 3x_1^2x_2 + 4x_2x_3^2 + x_1x_2x_3$$

$$\frac{\partial f}{\partial x_1} = 6x_1x_2 + x_2x_3$$
$$\frac{\partial f}{\partial x_2} = 3x_1^2 + 4x_3^2 + x_1x_3$$
$$\frac{\partial f}{\partial x_3} = 8x_2x_3 + x_1x_2$$
$$\nabla f = \begin{pmatrix} 6x_1x_2 + x_2x_3 \\ 3x_1^2 + 4x_3^2 + x_1x_3 \\ 8x_2x_3 + x_1x_2 \end{pmatrix}$$

Exercise 2.

1.

$$f(x_1, x_2, x_3) = x_1^{\ln(x_2^2+x_3)}$$
$$\frac{\partial f}{\partial x_1} = \ln(x_2^2 + x_3)x_1^{\ln(x_2^2+x_3)-1}$$

$$\frac{\partial f}{\partial x_2} = x_1^{\ln(x_2^2+x_3)} \cdot \ln x_1 \cdot \frac{1}{x_2^2 + x_3} \cdot 2x_2$$

$$\frac{\partial f}{\partial x_3} = x_1^{\ln(x_2^2+x_3)} \cdot \ln x_1 \cdot \frac{1}{x_2^2 + x_3}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \ln(x_2^2 + x_3)(\ln(x_2^2 + x_3) - 1)\, x_1^{\ln(x_2^2+x_3)-2}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{2x_2}{x_2^2 + x_3} x_1^{\ln(x_2^2+x_3)-1} + \ln(x_2^2 + x_3)\, x_1^{\ln(x_2^2+x_3)-1} \ln x_1 \cdot \frac{2x_2}{x_2^2 + x_3}$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{1}{x_2^2 + x_3} x_1^{\ln(x_2^2+x_3)-1} + \ln(x_2^2 + x_3)\, x_1^{\ln(x_2^2+x_3)-1} \cdot \ln x_1 \cdot \frac{1}{x_2^2 + x_3}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_2^2} &= x_1^{\ln(x_2^2+x_3)}(\ln x_1)^2 \cdot \frac{2x_2}{x_2^2+x_3} \cdot \frac{1}{x_2^2+x_3} \cdot 2x_2 && (1)\\
&\quad +x_1^{\ln(x_2^2+x_3)} \ln x_1 \frac{(x_2^2+x_3)\cdot 2 - 2x_2 \cdot 2x_2}{(x_2^2+x_3)^2} && (2)\\
&= \frac{x_1^{\ln(x_2^2+x_3)}}{(x_2^2+x_3)^2}\left((\ln x_1)^2 \cdot 4x_2^2 + 2(\ln x_1)(x_2^2+x_3) - \ln x_1 \cdot 4x_2^2\right) && (3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_3 \partial x_2} &= x_1^{\ln(x_2^2+x_3)} \ln x_1 \frac{1}{x_2^2+x_3} \ln x_1 \frac{2x_2}{x_2^2+x_3} + x_1^{\ln(x_2^2+x_3)} \ln x_1 \cdot \frac{2x_2(-1)}{(x_2^2+x_3)^2}\\
&= x_1^{\ln(x_2^2+x_3)} \ln x_1 \frac{2x_2}{(x_2^2+x_3)^2}(\ln x_1 - 1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_3^2} &= x_1^{\ln(x_2^2+x_3)} \ln x_1 \frac{1}{x_2^2+x_3} \ln x_1 \frac{1}{x_2^2+x_3} + x_1^{\ln(x_2^2+x_3)} \ln x_1 \frac{(-1)}{(x_2^2+x_3)^2}\\
&= x_1^{\ln(x_2^2+x_3)} \ln x_1 \frac{1}{(x_2^2+x_3)^2}(\ln x_1 - 1)
\end{aligned}
$$

2. $f(x_1, x_2) = x_1^2 x_2 + e^{x_1} x_2$

$$\frac{\partial f}{\partial x_1} = 2x_1x_2 + e^{x_1}x_2 \qquad \frac{\partial f}{\partial x_2} = x_1^2 + e^{x_1}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2x_2 + e^{x_1}x_2 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2x_1 + e^{x_1}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 2x_1 + e^{x_1} \qquad \frac{\partial^2 f}{\partial x_2^2} = 0$$

Exercise 3.

$$x_1(p_1,p_2) = 10 - p_1 + 2p_2\,, \quad x_2(p_1,p_2) = 8 + 2p_1 - 6p_2$$

Cost function: $\quad C(x_1,x_2) = 4x_1 + 2x_2$

Profit : $\quad P(p_1,p_2) = p_1x_1 + p_2x_2 - C(x_1,x_2)$

$$\begin{aligned}
\implies P(p_1,p_2) &= p_1(10-p_1+2p_2) + p_2(8+2p_1-6p_2) - 4(10-p_1+2p_2) - 2(8+2p_1-6p_2) \\
&= 10p_1 - p_1^2 + 2p_1p_2 + 8p_2 + 2p_1p_2 - 6p_2^2 - 40 + 4p_1 - 8p_2 - 16 - 4p_1 + 12p_2 \\
&= -p_1^2 - 6p_2^2 + 4p_1p_2 + 10p_1 + 12p_2 - 56
\end{aligned}$$

$$\frac{\partial P}{\partial p_1} = -2p_1 + 4p_2 + 10 \stackrel{!}{=} 0 \qquad (1)$$

$$\frac{\partial P}{\partial p_2} = -12p_2 + 4p_1 + 12 \stackrel{!}{=} 0 \qquad (2)$$

$$2\cdot(1)+(2) \implies -4p_2 + 32 = 0 \implies p_2 = 8$$

$$(1) \implies p_1 = 21$$

$P = (21,8)$ is a stationary point.

$$\frac{\partial^2 G}{\partial p_1^2} = -2, \quad \frac{\partial^2 G}{\partial p_2^2} = -12, \quad \frac{\partial^2 G}{\partial p_2 \partial p_1} = 4 \quad \text{for all } p_1, p_2$$

$$\frac{\partial^2 G}{\partial p_1^2}\cdot\frac{\partial^2 G}{\partial p_2^2} - \left(\frac{\partial^2 G}{\partial p_2\partial p_1}\right)^2 = (-2)\cdot(-12) - 4^2 = 24 - 16 = 8 > 0$$

for p_1, p_2

$\implies$ $P(p_1,p_2)$ has a local maximum at $x = (21,8)$. Since $\lim_{p_1\to\pm\inf} P(p_1,p_2) = -\inf$ and $\lim_{p_2\to\pm\inf} P(p_1,p_2) = -\inf$ this is also the global maximum.

Exercise 4.

1.

$$f(x,y) = e^{x^2-4x} + \frac{1}{3}y^2 - 2y$$

$$f_x(x,y) = e^{x^2-4x}(2x-4) \stackrel{!}{=} 0 \implies 2x - 4 = 0 \implies x = 2$$

$$f_y(x,y) = \frac{2}{3}y - 2 \stackrel{!}{=} 0 \implies y = 3$$

$(x_0, y_0) = (2,3)$ is a stationary point (i.e. a candidate for an extremum).

$$\begin{aligned} f_{xx}(x,y) &= (2x-4)e^{x^2-4x}(2x-4) + e^{x^2-4x} \cdot 2 \\ &= \left[(2x-4)^2 + 2\right] e^{x^2-4x} > 0 \quad \text{for all } x,\ y \end{aligned}$$

$$f_{yy}(x,y) = \frac{2}{3} > 0\,, \quad f_{xy}(x,y) = 0 \quad \text{for all } x,\ y\,,$$

$\det(H^f) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = ((2x-4)^2+2)e^{x^2-4x} \cdot \frac{2}{3} - 0 > 0$ für all $x,\ y \in \mathbb{R}$

Thus f has a local minimum at $(X_0, y_0) = (2,3)$.

2. $f(x,y) = x^3 - y^3 + 27x + 12y - 4$

   $f_x = 3x^2 + 27 = 0 \Rightarrow$ There is no real solution, i.e. there is no extremum.

3. $f(x,y) = x^3 - y^3 - 27x + 12y - 4$

$$f_x = 3x^2 - 27 \overset{!}{=} 0 \Longrightarrow x = \pm 3$$

$$f_y = -3y^2 + 12 \overset{!}{=} 0 \Longrightarrow y = \pm 2$$

$\Longrightarrow P_1 = (3,2),\ P_2 = (3,-2),\ P_3 = (-3,2),\ P_4 = (-3,-2)$ are stationary points.

$$f_{xx} = 6x\,, \qquad f_{yy} = -6y\,, \qquad f_{xy} = f_{yx} = 0$$

$$\det(H^f) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = -36xy$$

- $P_1 = (3,2)$

$$\det(H^f(3,2)) = -36 \cdot 3 \cdot 2 < 0$$

$\Longrightarrow$ There is no extremum at $P_1 = (3,2)$ (We call this *saddlepoint*).

- $P_2 = (3,-2)$

$$\det(H^f((3,-2)) = -36 \cdot 3 \cdot (-2) > 0$$

$$f_{xx} = 6 \cdot 3 = 18 > 0$$

$\Longrightarrow$ There is a local minimum at $P_2 = (3,-2)$.

- $P_3 = (-3, 2)$

$$\det(H^f((-3,2)) = -36 \cdot (-3) \cdot 2 > 0$$
$$f_{xx} = 6 \cdot (-3) = -18 < 0$$

$\Longrightarrow$ There is a local maximum at $P_3 = (-3, 2)$.

- $P_4 = (-3, -2)$

$$\det(H^f((-3,-2)) = -36 \cdot (-3) \cdot (-2) < 0$$

$\Longrightarrow$ There is a saddlepoint (no extremum) at $P_4 = (-3, -2)$.

Exercise 5.

1.
$$\frac{\partial f}{\partial x_1} = 2x_1 e^{x_2^2} \implies \frac{\partial f}{\partial x_1}(1,0) = 2$$
$$\frac{\partial f}{\partial x_2} = x_1^2\, 2x_2 e^{x_2^2} \implies \frac{\partial f}{\partial x_2}(1,0) = 0$$

2. (a)

$$\begin{aligned} df|_{x=x^{(0)}} &= \frac{\partial f}{\partial x_1}(1,0) + \frac{\partial f}{\partial x_1}(1,0) = 2 + 0 = 2 \\ f(1.1, 0.1) &= f((1,0) + (0.1, 0.1)) \\ &\approx f(1,0) + df((0.1,0.1),(1,0)) = 1 + 2\cdot 0.1 + 0 \cdot 0.1 = 1.2 \end{aligned}$$

Exercise 6.

$$f(x_1, x_2, x_3) = x_1 x_2^2 + x_2 \ln x_3\,, \quad r = (r_1,\ r_2,\ r_3)$$

$$\begin{aligned} D_v f(x^0) &= \frac{1}{||v||} \sum_{i=1}^{3} \left.\frac{\partial f}{\partial x_i}\right|_{x=x^0} \cdot v_i \\ &= \frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \left((x_2^0)^2 v_1 + (2x_1^0 x_2^0 + \ln x_3^0) v_2 + \frac{x_2^0}{x_3^0} v_3 \right) \end{aligned}$$

$$x^0 = (1,3,5), \quad v^{(1)} = (1,3,4), \quad v^{(2)} = (0,1,0)$$

1.
$$v^{(1)} = (1,3,4), \quad x^0 = (1,3,5)$$
$$\implies D_{(1,3,4)} f(1,3,5) = \frac{1}{\sqrt{26}} \left(3^2 \cdot 1 + (2\cdot 1 \cdot 3 + \ln 5) 3 + \frac{3}{5} \cdot 4 \right) \approx 6.7127$$

2.
$$v^{(2)} = (0,1,0), \quad x^0 = (1,3,5)$$
$$\implies \underbrace{D_{(0,1,0)} f(1,3,5)}_{\left.\frac{\partial f}{\partial x_2}\right|_{x=x^0}} = (2 \cdot 1 \cdot 3 + \ln 5) \cdot 1 \approx 7.6094$$